A Continuous Spatial Choice Logit Model of a Polycentric City

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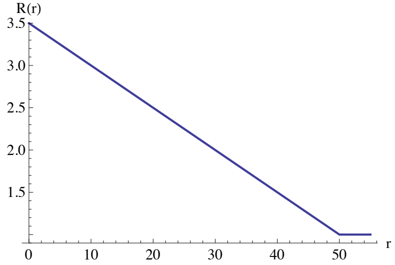
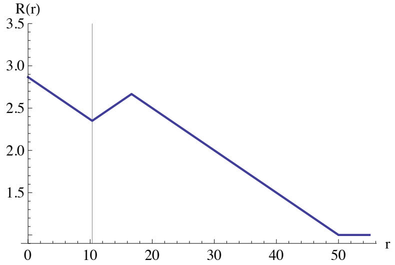
The classical monocentric city model

- Alonso (1964), Mills (1967), Muth (1969)
- Assumption: exogenously determined central business district (CBD)
- Mechanism: households optimally trade off commuting costs and land prices
- Prediction: negatively sloped land-price gradient
- Land rent for a linear city and uniform land use:



Polycentric cities in the classic model

- Polycentric cities can be considered unions of monocentric cities in which workers select the business district as to maximize wages minus commuting costs



- Predictions:
    - Complete residential segregation of employees of different business districts
    - No cross-commuting
    - Symmetric land rent schedule around secondary business districts (SBDs)

Conflicting empirical evidence

- Cross commuting / excess commuting in cities (Hamilton, 1982; Small & Song, 1992)
- Workplace accessibility effect (Adair et al., 2000; Osland & Thorsen, 2008; Ahlfeldt, 2011)
- Asymmetry of the land price schedule around subcenters (Ahlfeldt et al., 2012)

A discrete choice model

- Anas (1990) considered monocentric city households where utility functions contained additive idiosyncratic utility constants that differed among households. This type of random taste heterogeneity implies that household choices can only be probabilistically determined.
- In partitioning a metropolitan area into a large number of smaller areas, he applied a discrete choice model (McFadden, 1973, 1976, 1978) and used a multinomial logit model to describe the location choices of households.
- Assuming that common utilities do not differ across locations, the multinomial logit model in discrete space asymptotically converges to the Alonso-Mills-Muth model with one type of household.
- Extensions to allow for polycentricity and consumption and production externalities: Anas & Kim, 1996; Anas & Xu, 1999; Anas & Rhee, 2006; Tscharaktschiew & Hirte, 2010

Limitations of the discrete choice model

- Typically analytically intractable and requires extensive simulations
- Is not easily analyzed with geometric tools
- Not directly related to the Alonso-Muth-Mills model

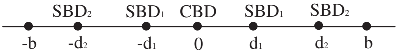
A continuous choice model

- Continuous logit model (McFadden, 1976; Ben-Akiva & Watanatada, 1981; Ben-Akiva et al., 1985)
- Probabilistic choice model for a continuum of alternatives
- Probabilistic analog of the classic Alonso-Muth-Mills model

Predictions of the continuous choice model in this paper

- Incomplete residential segregation of employees of different business districts
- Extensive cross-commuting
- Workplace accessibility effects: non-linear effects of wages and accessibility
- Asymmetric land rent schedule around SBDs
- Farming in fringe areas of the city: in some parts of the city, only a fraction of the available land is used by city dwellers and the land rent does not vary with distance to the city center because it is already equal to the land opportunity costs.

The model



- Linear city: land use along a straight line with unit width
- The city hosts total population N and spans from $-b$ to b, with b exogenously given.
- $2n+1$ business districts, where district k is located at d_k: $d_k < d_{k+1}$, $k = -n, \ldots, n$.
- CBD at $d_0 = 0$ and $2n$ SBDs
- Exogenously given wage at workplace k: w_k
- Symmetry around the CBD: $d_k = -d_{-k}$ and $w_k = w_{-k}$
- Residence location: $r \in [-b, b]$
- Linear round-trip commuting costs: $T = t|r - d_k|$
- Land opportunity costs: R_A

Choices

- Every citizen commutes to a workplace of his/her own choice,
- inelastically supplies one unit of labor,
- earns only labor income,
- consumes the numéraire good at price 1,
- and consumes one unit of land ($\approx$ housing) at a location of his/her own choice.

Utility

- Total probabilistic utility consists of the deterministic utility and an idiosyncratic location taste constant:

$$v(r,k) = u(r,k) + \epsilon(r,k) \text{ for all } r \in [-b,b], k \in \{-n,\dots,n\}.$$

- Deterministic linear utility component:

$$u(r,k) = w_k - t|r - d_k| - R(r)$$

- $\epsilon(r,k)$ is a random variable that is independently and identically (i.i.d) Gumbel distributed with mean zero and variance $\sigma^2 = \pi^2/6\lambda^2$ for all choice sets, where $\lambda > 0$ is the dispersion parameter of the distribution.
- A smaller λ indicates a higher diversity of preferences across consumers who react differently to the same deterministic-utility schedule. If $\lambda \to 0$, individuals choose randomly; if $\lambda \to \infty$, individuals only select locations where they achieve the maximum

Location choice

- Affordability assumption: $w_k \geq t|r - d_k| + R(r)$ everywhere
- Necessary for affordability: $w_k > R_A + t(d_k + b)$
- Logit choice density function:

$$p(r,k) = \frac{e^{\lambda u(r,k)}}{P}, \text{ where } P := \int_{-b}^{b} \sum_{k=-n}^{n} e^{\lambda u(s,k)} ds > 0,$$

$$r \in [-b, b], k \in \{-n, \ldots, n\}.$$

Land use

- Urban land demand at location r

$$L(r) = N \sum_{k=-n}^{n} p(r,k)\,, \quad \text{where } L(r) \geq 0\,.$$

Definition (Spatial urban equilibrium)

For uniform land use in a symmetric polycentric city with an exogenously determined border, b, and a set of strictly positive exogenously determined wages, $\{w_{-n}, \dots, w_n\}$, where $w_{-k} = w_k \geq t|r - d_k| + R(r)$, for $k = 0, \dots, n$, a spatial urban equilibrium is defined as a land rent function, $R(r)$, with $R \in \Re_0^+$, for $r \in [-b, b]$, with the following properties: All agents take all wages, the continuum of land prices, and the consumption good price normalized to 1 as given; all citizens select locations and workplaces according to the logit choice density function; each citizen lives somewhere and works in some business district within the boundaries of the city; at every location, r, landowners maximize revenue, and total land demand equals supply.

- In a spatial urban equilibrium, land demand equals available land at each location in the city.

Focus on a particular type of equilibrium

Definition (Spatial urban equilibrium with urban core and full affordability)

A symmetric spatial urban equilibrium with urban core is defined as a spatial urban equilibrium where

(a) $R(-r) = R(r)$ *and* $L(-r) = L(r)$, *for* $r \in [-b, b]$,

(b) and there is a critical location, $\bar{r} \leq b$, *such that*

Distance to the CBD	$r \in [0, \bar{r})$	$r = \bar{r}$	$r \in (\bar{r}, b]$
Urban land use	$L(r) = 1$	$L(r) = 1$	$L(r) < 1$
Land rent	$R(r) > R_A$	$R(r) = R_A$	$R(r) = R_A$

Equilibrium: properties

Proposition

If a symmetric spatial urban equilibrium with urban core exists such that the critical location, $\bar{r}$, satisfies

$$\bar{r} = \frac{N}{2} - \frac{1 + \mathcal{W}\left[-e^{-1-\lambda t(b-N/2)}\right]}{\lambda t} \in \left(\frac{N}{2} - \frac{1}{\lambda t}, \frac{N}{2}\right], \qquad (1)$$

where $\mathcal{W}$ is the principal value of the Lambert W function, it has the following properties:

(i) Outside location $\bar{r}$, urban land use is a declining exponential function of distance defined by

$$L(r) = e^{\lambda t(\bar{r}-r)} < 1, \quad \text{for } r \in (\bar{r}, b].$$

Proposition

(ii) In the area between the outermost SBD and location $\bar{r}$, the land rent is a decreasing linear function of distance defined by

$$R(r) = R_A + t(\bar{r} - r), \quad \textit{for } r \in [d_n, \bar{r}].$$

(iii) In the area between the CBD and the outermost SBD, the land rent is a non-linear function of distance defined

$$R(r) = R_A + t(\bar{r} - r) + \frac{1}{\lambda} \ln \left[\frac{\sum_{j=-n}^{i} e^{\lambda(w_j + td_j)} + \sum_{j=i+1}^{n} e^{\lambda[w_j - t(d_j - 2r)]}}{\sum_{j=-n}^{n} e^{\lambda(w_j + td_j)}} \right.$$

$$\textit{for } r \in [0, d_n), \textit{ and } d_j \leq r \textit{ for } j = 0, \ldots, i, \textit{ and } d_j > r \textit{ for } j = i + 1, \ldots$$

with

$$-t < \frac{dR(r)}{dr} < t.$$

Proposition

(iv) The land rent schedule is asymmetric around SBDs. For SBDs far away from the CBD, the land rent decreases more rapidly in the direction of the city's boundary than in the direction of the city center.
(v) An increase in the wage at the CBD or in one of the SBDs does not affect the critical location, $\bar{r}$, or the land rent in the outer area $[d_n, b]$. An increase in the wage in business district j, $0 \leq j < n$, increases the land rent in the area $[d_j, d_n)$,
but the increase may decrease the land rent in some parts of the area $[0, d_j)$.
(vi) If diversity tends to disappear, i.e., if λ goes to infinity, the critical location, $\bar{r}$, converges to $N/2$ and, for $r > \bar{r}$, urban land use, $L(r)$, converges to 0.

Equilibrium: existence

Proposition

If $b \geq N/2$*,* $N/2 - \left\{1 + \mathcal{W}\left[-e^{-1-\lambda t(b-N/2)}\right]\right\}/\lambda t > d_n$*,* $w_0 \geq w_0^-$*, and* $w_k > R_A + t(b + d_k)$*,* $k = 1, \ldots, n$*, a symmetric spatial urban equilibrium with urban core exists where the critical location,* $\bar{r}$*, satisfies Equation (1).*

Urban land use

- $n = 3,\ R_A = 1,\ t = 1,\ N = 10,\ b = 7.5,\ d_1 = 1,\ d_2 = 2,\ d_3 = 3$
- $\bar{r} = 4.03$
- $\lambda = 1$

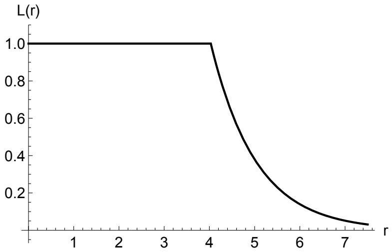


Figure : urban land use

Asymmetric wage distribution

- $w_0 = 22 > 20 = w_1 = w_2 = w_3$
- $\lambda = 1$

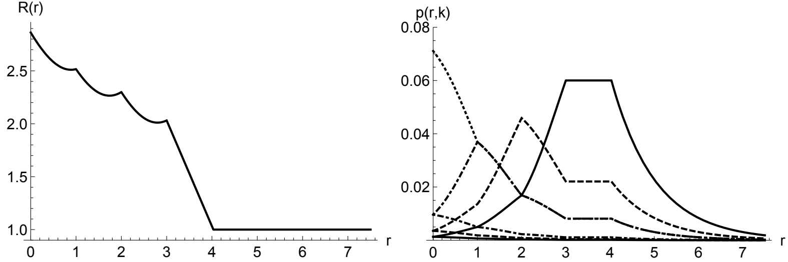


Figure : land rent (left chart) and logit choice density functions (right chart)

- $p(r,0)$: dotted, $p(r,d_1)$ and $p(r,-d_1)$: dot-dashed, $p(r,d_2)$ and $p(r,-d_2)$: dashed, $p(r,d_3)$ and $p(r,-d_3)$: solid.

Symmetric wage distribution

- $w_0 = 20 = w_1 = w_2 = w_3$
- $\lambda = 1$

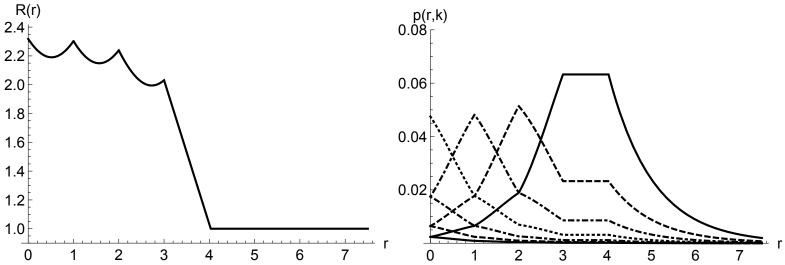


Figure : land rent (left chart) and logit choice density functions (right chart)

- $p(r, 0)$: dotted, $p(r, d_1)$ and $p(r, -d_1)$: dot-dashed, $p(r, d_2)$ and $p(r, -d_2)$: dashed, $p(r, d_3)$ and $p(r, -d_3)$: solid.

Rather homogenous population

- Symmetric wage distribution: $w_0 = 20 = w_1 = w_2 = w_3$
- $\lambda = 10$

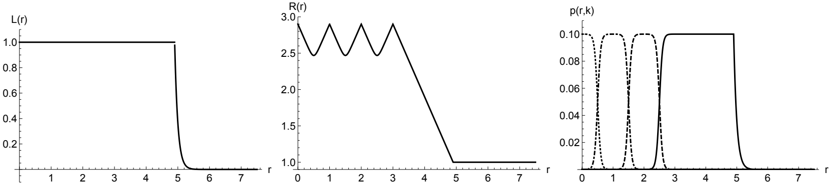


Figure : urban land use (left chart), land rent (middle chart), and logit choice density functions (right chart)

- $p(r,0)$: dotted, $p(r,d_1)$ and $p(r,-d_1)$: dot-dashed, $p(r,d_2)$ and $p(r,-d_2)$: dashed, $p(r,d_3)$ and $p(r,-d_3)$: solid.

Alternative equilibria

- If wages are generally low, living in some areas with high land rents may be unaffordable for workers employed at workplaces located relatively far away. Low wages may reduce the variance in the city's land rents.
- If wages in the center are low and wages in the SBDs in the outer areas of the city are high and/or if the outermost SBD is close to the city's boundary (i.e., if $N/2 - \left\{1 + \mathcal{W}\left[-e^{-1-\lambda t(b-N/2)}\right]\right\}/\lambda t < d_n$), then the area $[-d_n, d_n]$ may be partially used by farmers.

Open questions

- Multiple income classes: for a monocentric city, Wrede (2013)
- Non-linear utility with endogenous housing: for a monocentric city, Wrede (2013)
- Externalities and amenities
- Spatial pattern other than linear city

- Thank's for your attention !

Land use

$$L(r) = \frac{N}{e^{\lambda R(r)}\, P} \left[\sum_{k=-n}^{i} e^{\lambda[w_k - t(r-d_k)]} + \sum_{k=i+1}^{n} e^{\lambda[w_k - t(d_k - r)]} \right].$$

- i satisfies: $d_k \leq r$ for $k = -n, \ldots, i$, and $d_k > r$ for $k = i+1, \ldots, n$